SOLUTIONS OF THE FRACTIONAL REACTION EQUATION AND THE FRACTIONAL DIFFUSION EQUATION

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Abstract. In view of the role of reaction equations in physical problems, the authors derive the explicit solution of a fractional reaction equation of general character, that unifies and extends earlier results. Further, an alternative shorter method based on a result developed by the authors is given to derive the solution of a fractional diffusion equation. Fox functions and Mittag-Leffler functions are used for closed-form representations of the solutions of the respective differential equations.

1   Introduction

Fractional reaction and diffusion equations involve fractional derivatives with respect to time and space and are studied to describe anomalous reaction and diffusion of dynamical systems with chaotic motion. Fractional reaction equation for Hamiltonian chaos is discussed by Zaslavsky (1994). Solutions and applications of reaction equations are studied by Saichev and Zaslavsky (1997). Solutions of a fractional reaction equation is investigated by Haubold and Mathai (2000) for a simple production-destruction mechanism. This equation was generalized by Saxena, Mathai, and Haubold (2002). In recent articles, Saxena, Mathai, and Haubold (2002, 2004a, 2004b) discussed the solution of a number of generalized fractional reaction equations. In the present article, we investigate the solution of a unified fractional reaction equation, which provides unification and extension of results on fractional reaction equations given earlier by Haubold and Mathai (2000) and Saxena, Mathai, and Haubold (2002, 2004a). We also

present the solution of a fractional integral equations discussed by Miller and Ross (1993). Further, an alternative proof of the solution of a fractional diffusion equation given earlier by Kochubei (1990) is investigated, which is based upon a result given by Saxena, Mathai, and Haubold (2006). Most of the results are obtained in terms of generalized Mittag-Leffler functions in elegant and compact forms, which are also suitable for numerical computation.

The paper is organized as follows. Section 2 provides the solution of a unified fractional reaction equation while Section 3 considers special cases of the equation. A shorter alternative method for the solution of a fractional diffusion equation discussed earlier by Kochubei (1990) is presented in Section 4. A series representation and asymptotic expansion of the solution are given in Section 5. An H-function representation of a one-sided Lévy stable density is also obtained.

2   Fractional reaction equation

In this Section, we present a method based on Laplace transform for deriving the solution of the unified fractional reaction equations.

Theorem 1. If $Re(\nu_j) > 0, a_j > 0, j \in \mathrm{N}$, and f(t) be a given function, defined on $\Re_+$, then the equation

$$N(t) - N_0 f(t) = -\sum_{j=1}^{n} a_j \, _0D_t^{-\nu_j} N(t), \tag{1}$$

is solvable and its particular solution is given by

$$
\begin{aligned}
N(t) \;=\; & N_0 \sum_{l=0}^{\infty} (-1)^l \sum_{r_1+\ldots+r_{n-1}=l} \frac{(l)!}{(r_1)! \ldots (r_{n-1})!} \left\{ \prod_{\mu=1}^{n-1} (a_{\mu+1})^{r_\mu} \right\} \\
& \int_0^t f(u)(t-u)^{\sum_{\mu=1}^{n-1} \nu_{\mu+1}-1} E_{\nu_1, \sum_{\mu=1}^{n-1} \nu_{\mu+1}}^{(l+1)} [-a_1(t-u)^{\nu_1}] du, \quad (2)
\end{aligned}
$$

where the summation in (2) is taken over all nonnegative integers $r_1, \ldots, r_n$ such that $r_1 + \ldots + r_{n-1} = l$, and provided that the series and integral in (2) are convergent. Here $_0D_t^{-\nu_j}, j \in \mathrm{N}$ are Riemann-Liouville fractional integrals, defined by

$$_0D_t^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-u)^{\nu-1} f(u) du, \, Re(\nu) > 0, \tag{3}$$

with $_0D_t^0 f(t) = f(t)$ (Oldham and Spanier, 1974; Miller and Ross, 1993), $E_{\beta,\gamma}^{\delta}(z)$ is the generalized Mittag-Leffler function, defined by Prabhakar (1971)

in terms of series representation as

$$E^{\delta}_{\beta,\gamma}(z) = \sum_{\tau=0}^{\infty} \frac{(\delta)_{\tau} z^{\tau}}{\Gamma(\beta\tau + \gamma)(\tau)!} \quad (\beta, \gamma, \delta \in C, Re(\beta) > 0, Re(\gamma) > 0). \tag{4}$$

Proof. By the application of the convolution theorem of the Laplace transform (Erdélyi et al., 1953, p. 259) to (3), we find that

$$\begin{aligned} L\left\{ {}_0 D_t^{-\nu} f(t); s \right\} &= L\left\{ \frac{t^{\nu-1}}{\Gamma(\nu)} \right\} L(f(t)), \\ &= s^{-\nu} f^{\sim}(s), \end{aligned} \tag{5}$$

where $f^{\sim}(s) = \int_0^{\infty} e^{-st} f(t) dt, s \in C, Re(s) > 0$. Applying Laplace transform to (1) and using (5), it gives

$$\begin{aligned} N^{\sim}(s) &= \frac{N_0 f^{\sim}(s)}{1 + a_1 s^{-\nu_1} + \ldots + a_n s^{-\nu_n}} \tag{6} \\ &= N_0 f^{\sim}(s) \sum_{l=0}^{\infty} (-1)^l \frac{\left(\sum_{j=1}^{n-1} a_{j+1} s^{-\nu_{j+1}} \right)^l}{(1 + a_1 s^{-\nu_1})^{l+1}}, \sum_{j=2}^{n} a_j s^{-\nu_j} < 1 + a_1 s^{-\nu_1}. \end{aligned}$$

If we employ the identity (Abramowitz and Stegun, 1968, p. 823)

$$(x_1 + \ldots + x_m)^l = \sum_{r_1 + \ldots + r_n = l} \frac{(l)!}{(r_1)! \ldots (r_n)!} \prod_{\mu=1}^{m} x_{\mu}^{r_{\mu}}, \tag{7}$$

where the summation is taken over all nonnegative integers, $r_1, \ldots, r_n$, such that $r_1 + \ldots + r_n = l$, then for $|a_1 s^{-\nu_1}| < 1$, (7) transforms into the form

$$\begin{aligned} N^{\sim}(s) &= N_0 f^{\sim}(s) \sum_{l=0}^{\infty} (-1)^l \sum_{\substack{r_1 + \ldots + r_{n-1} = l \\ r_1 > \ldots r_{n-1} > 0}} \frac{(l)!}{(r_1)! \ldots (r_{n-1})!} \\ &\quad \frac{\left\{ \prod_{\mu=1}^{n-1} (a_{\mu+1})^{r_{\mu}} \right\} s^{-\sum_{\mu=1}^{n-1} \nu_{\mu+1}}}{(1 + a_1 s^{-\nu_1})^{l+1}}. \end{aligned} \tag{8}$$

Taking the inverse Laplace transform of (8) by making use of the formula (Kilbas, Saigo, and Saxena, 2004, eq. (12))

$$L^{-1}\left\{ s^{-\gamma} (1 - a s^{-\beta})^{-\delta}; t \right\} = t^{\gamma-1} E^{\delta}_{\beta,\gamma}(a t^{\beta}), \tag{9}$$

where $Re(s) > |a|^{1/Re(\gamma)}, Re(\gamma) > 0, Re(s) > 0$, and applying the convolution theorem of the Laplace transform, the result (2) is established.

Remark 1. The generalized Mittag-Leffler function defined by (4) is studied by Prabhakar (1971) and Kilbas, Saigo, and Saxena (2004). Recently this function is used in the theory of finite-size scaling of systems with strong anisotropy and long-range interaction by Chamati and Tonshev (2006).

3 Special cases

Some special cases of Theorem 1 are of interest to be highlighted. If we set $\nu_j = j\nu, a_j = \binom{n}{j}c^{j\nu}(j \in \mathrm{N})$, we obtain

Theorem 2. If $Re(\nu) > 0, c > 0$ and $f(x) \in \Re_+$, then the equation

$$N(t) - N_0 f(t) = -\sum_{r=1}^{n} \binom{n}{r} c^{\nu r} D_t^{-\nu r} N(t), \tag{10}$$

is solvable and its solution has the form

$$N(t) = N_0 \frac{d}{dt} \int_0^t f(u) E_{\nu,1}^n[-c^\nu (t-u)^\nu] du, \tag{11}$$

where $E_{\nu,1}^n(x)$ is the generalized Mittag-Leffler function defined by (4) and provided that the integral (11) is convergent.

When $n = 1$, we obtain the following result given by Hille and Tamarkin (1930).

Corollary 2.1. Let $Re(\nu) > 0, c > 0$ and let $f(x) \in \Re_+$, then for the solution of the integral equation

$$N(t) - N_0 f(t) = -c^\nu \,_0D_t^{-\nu} N(t), \tag{12}$$

holds the following formula

$$N(t) = N_0 \frac{d}{dt} \int_0^t f(u) E_\nu[-c^\nu (t-u)^\nu] du, \tag{13}$$

where $E_\nu(z)$ is an entire function of order $\rho = \frac{1}{\nu}$ and type $\sigma = 1$, defined by

$$E_\nu(z) = \sum_{\mu=1}^{\infty} \frac{z^\mu}{\Gamma(\mu\nu + 1)}, \quad (\nu \in C, Re(\nu) > 0). \tag{14}$$

Note 1. The above result has also been given by the authors in a different form (Saxena, Mathai, and Haubold, 2004a, 2004b).

If we set $f(t) = t^{\gamma-1} E_{\nu,\gamma}^\delta[-(ct)^\nu]$, Theorem 2 yields

Corollary 2.2. Let $Re(\nu) > 0, Re(\gamma) > 0, c > 0$, then for the solution of the reaction equation

$$N(t) - N_0 t^{\gamma-1} E_{\nu,\gamma}^\delta[-(ct)^\nu] = -\sum_{r=1}^{n} \binom{n}{r} c^{r\nu} \,_0D_t^{-r\nu} n \in \mathrm{N} \tag{15}$$

holds the relation

$$N(t) = N_0 t^{\gamma-1} E_{\nu,\gamma}^{\delta+n}[-(ct)^\nu], n \in \mathrm{N}. \tag{16}$$

For $f(t) = t^{\rho-1}$, Theorem 2 yields the following result

Corollary 2.3. If $Re(\rho) > 0, Re(\nu) > 0, c > 0$, then for the solution of the equation

$$N(t) - N_0 t^{\rho-1} = -\sum_{r=1}^{n} \binom{n}{r} c^{r\nu} \, {}_0D_t^{-r\nu} N(t), r \in \mathrm{N}, \tag{17}$$

holds the relation

$$N(t) = N_0 t^{\rho-1} \Gamma(\rho) E_{\nu,\rho}^n [-(ct)^\nu], r \in \mathrm{N}. \tag{18}$$

For $n = 1$, eq. (18) reduces to a result given by Saxena, Mathai, and Haubold (2002, p. 283, eq. (15)). When $a_j = a^j s^{\nu j}$, for $j = 1, \ldots, \nu$, we obtain

Theorem 3. Let $Re(\nu) > 0, a > 0, t > 0, n > 1, |a^{n+1} s^{-(n+1)\nu}| < 1$, and $f(x)$ be a given function defined on $\Re_+$, then the equation

$$N(t) - N_0 f(t) = -\sum_{r=1}^{n} a^r \, {}_0D_t^{-\nu r} N(t), \tag{19}$$

is solvable and its solution is given by

$$N(t) = N_0 \left\{ \frac{d}{dt} \int_0^t f(u) E_{(n+1)\nu,\nu}[a^{n+1}(t-u)^{(n+1)\nu}] du \right.$$
$$\left. -a \int_0^t (t-u)^{\nu-1} E_{(n+1)\nu,\nu}[a^{n+1}(t-u)^{(n+1)\nu}] du \right\}, \tag{20}$$

where $E_{(n+1)\nu,\nu}(z)$ is the generalized Mittag-Leffler function $E_{\alpha,\beta}(z)$ defined as

$$E_{\alpha,\beta}(z) = \sum_{\mu=1}^{\infty} \frac{z^\mu}{\Gamma(\alpha\mu + \beta)}, \quad (\alpha, \beta \in C, Re(\alpha) > 0, Re(\beta) > 0) \tag{21}$$

and provided that the integral in (20) is convergent.

If we take $\nu^j = j\nu$, for $j = 1, \ldots n$, then it is interesting to note that Theorem 1 yields the following result given by (Miller and Ross, 1993) in a different form:

Theorem 4. Let $Re(\nu) > 0, a_j > 0$, and $f(x)$ be a given function defined on $\Re_+, |a_1 s^{-\nu}| < 1$, then the fractional reaction equation

$$N(t) - N_0 f(t) = -\sum_{j=1}^{n} a_j \, {}_0D_t^{-j\nu} N(t), \tag{22}$$

is solvable and has the solution given by

$$N(t) = N_0 \sum_{l=0}^{\infty} (-1)^l \sum_{r_1+\ldots+r_{n-1}=l} \frac{(l)!}{(r_1)!\ldots(r_{n-1})!} \left\{ \prod_{\mu=1}^{n-1} (a_{\mu+1})^{r_\mu} \right\} \tag{23}$$
$$\times \int_0^t f(u)(t-u)^{\sum_{\mu=1}^{n-1} \nu(\mu+1)r_\mu - 1} E_{\nu_1, \sum_{\mu=1}^{n-1} \nu(\mu+1)r_\mu}^{(l+1)} [-a_1(t-u)^{\nu_1}] du,$$

provided that the series and integral in (23) are convergent.

5

4 Fractional diffusion equation

In this Section we present an alternative shorter method for deriving the solution of a diffusion equation discussed earlier by Kochubei (1990).

Theorem 5. Consider the Cauchy problem

$$_0D_t^\alpha N(x,t) = -c^\nu \Delta N(x,t), \ (0 < \alpha < 1; x \in \Re^n; \ 0 < t \leq T), \qquad (24)$$

with

$$N(x, t=0) = \delta(x), x \in \Re, \lim_{x \to \pm\infty} N(x,t) = 0 \qquad (25)$$

$_0D_t^\alpha$ is the regularized Caputo (1969) partial fractional derivative with respect to t, defined by

$$_0D_t^\alpha N(x,t) = \frac{1}{\Gamma(1-\alpha)} \left[\frac{\partial}{\partial t} \int_0^t \frac{N(x,s)ds}{(t-x)^\alpha} - \frac{N(x,0)}{t^\alpha} \right],$$

and Δ is the Laplacian. The fundamental solution of the above Cauchy problem is given by

$$N(x,t) = |x|^{-n} \pi^{-\frac{n}{2}} H_{1,2}^{2,0} \left[\frac{|x|^2 t^{-\alpha}}{4c^\nu} |_{(n/2,1),(1,1)}^{(1,\alpha)} \right], \qquad (26)$$

where $H_{1,2}^{2,0}(.)$ is the H-function (Mathai and Saxena, 1978).

Proof. Applying the Laplace transform with respect to t, using the result (Caputo, 1969)

$$L\{_0D_t^\alpha N(x,t)\} = s^\alpha N^\sim(x,s) - \sum_{r=0}^{m-1} s^{\alpha-r-1} N^{(r)}(x,0),$$

$$m - 1 < \alpha \leq m, \ m \in \mathrm{N},$$

and Fourier transform with respect to x, gives

$$s^\alpha N^{\sim^*}(k,s) - s^{\alpha-1} = -c^\nu |k|^2 N^{\sim^*}(k,s),$$

where the symbol "$\sim$" indicates the Laplace transform with respect to the time variable t and the symbol "$*$" the Fourier transform with respect to the space variable x.
Solving for $N^{\sim^*}$, we have

$$N^{\sim^*}(k,s) = \frac{s^{\alpha-1}}{s^\alpha + c^\nu |k|^2}. \qquad (27)$$

By virtue of the following Fourier transform formula (Samko, Kilbas, and Marichev, 1990, p. 538, eq. (27.1))

$$\left(F_x\left[|x|^{(2-n)/2}K_{(n-2)/2}(a|x|)\right]\right)(\tau) = \left(\frac{2\pi}{a}\right)^{n/2}\frac{a}{a^2+\tau^2}, (\tau \in \Re^n; n \in \mathbb{N}, a > 0),$$
(28)

where the multidimensional Fourier transform with respect to $x \in \Re^n$ is defined by

$$(F_xN)(\tau,t) = \int_{\Re^n} N(x,t)e^{ix\tau}dx \ (\tau \in \Re^n; t > 0)$$
(29)

and $K_\nu(.)$ is the modified Bessel function of the second kind, yields

$$\tilde{N}(x,s) = c^{-\nu}s^{\alpha-1}(2\pi)^{-\frac{n}{2}}\left(\frac{|x|c^{\frac{\nu}{2}}}{s^\alpha 2}\right)^{1-\frac{n}{2}}K_{n-2/2}\left[\frac{|s^{\frac{\alpha}{2}}|x|}{c^{\frac{\nu}{2}}}\right].$$
(30)

In order to invert the Laplace transform, we employ the following result given by the authors (Saxena, Mathai, and Haubold, 2006)

$$L^{-1}\left\{s^{-\rho}K_\nu(zs^\sigma);t\right\} = \frac{1}{2}t^{\rho-1}H_{1,2}^{2,0}\left[\frac{z^2t^{-2\sigma}}{4}\left|_{(\frac{\nu}{2},1)(-\frac{\nu}{2},1)}^{(\rho,2\sigma)}\right.\right],$$
(31)

where $K_\nu(x)$ is the modified Bessel function of the second kind, $Re(z^2) > 0, Re(s) > 0$. Thus we obtain the solution in the form

$$N(x,t) = \frac{1}{2}(2\pi)^{-\frac{n}{2}}c^{-\frac{\nu}{2}-\frac{n\nu}{4}}|x|^{1-\frac{n}{2}}t^{-\frac{\alpha}{2}-\frac{\alpha n}{4}}s^{\frac{\alpha}{2}+\frac{\alpha n}{4}-1}$$

$$H_{1,2}^{2,0}\left[\frac{t^{-\alpha}|x|^2}{4c^\nu}\left|_{(\frac{n-2}{2},1),(\frac{2-n}{4},1)}^{(1-\frac{\alpha}{2}-\frac{\alpha n}{4},\alpha)}\right.\right].$$
(32)

By virtue of a result in Mathai and Saxena (1978),

$$x^\sigma H_{p,q}^{m,n}\left[x|_{(b_q,b_q)}^{(a_p,a_p)}\right] = H_{p,q}^{m,n}\left[x|_{(b_q+\sigma B_q,B_q)}^{(a_p+\sigma A_p,A_p)}\right],$$
(33)

the power of the expression $[\{t^{-\nu}|x|^2\}/4c^\nu]$ can be absorbed inside the H-function and consequently we obtain

$$N(x,t) = |\pi^{\frac{1}{2}}x|^{-n}H_{1,2}^{2,0}\left[\frac{t^{-\alpha}|x|^2}{4c^\nu}\left|_{(\frac{n}{2},1),(1,1)}^{(1,\alpha)}\right.\right].$$
(34)

Remark 1. If we employ the identity (Mathai and Saxena, 1978)

$$H_{p,q}^{m,n}\left[x^\lambda|_{(b_q,B_q)}^{(a_p,A_p)}\right] = \frac{1}{\lambda}H_{p,q}^{m,n}\left[x|_{(b_q,B_q/\lambda)}^{(a_p,A_p/\lambda)}\right], \lambda > 0$$
(35)

the solution given by (32) can be expressed in the form

$$N(x,t) = \frac{1}{\alpha}|\pi^{\frac{1}{2}}x|^{-n}H_{1,2}^{2,0}\left[\frac{t^{-1}|x|^{2\alpha}}{(4c^\nu)^{\frac{1}{\alpha}}}\left|_{(\frac{n}{2},\frac{1}{\alpha}),(1,\frac{1}{\alpha})}^{(1,1)}\right.\right],$$
(36)

where $\alpha > 0$. We also note that the above form of the solution is due to Schneider and Wyss (1989). There is one importance of our result (32) that it includes the Lévy stable density in terms of the H-function as shown in (34). Similarly, using the identity (33) we arrive at

$$\frac{1}{2}\pi^{\frac{1}{2}}|x|^{-n}H_{1,2}^{2,0}\left[\frac{t^{-\frac{\alpha}{2}}|x|}{2c^{\frac{\nu}{2}}}\bigg|_{(\frac{n}{2},\frac{1}{2}),(1,\frac{1}{2})}^{(1,\frac{\alpha}{2})}\right], \tag{37}$$

where n is not an even integer. This form of the H-function is useful in determining its expansion in powers of x. Due to importance of the solution, we also discuss its series representation and behavior.

5 Series representation of the solution

Using the series expansion for the H-function given in Mathai and Saxena (1978), it follows that

$$H_{1,2}^{2,0}\left[x\bigg|_{(\frac{n}{2},\frac{1}{\alpha}),(1,\frac{1}{\alpha})}^{(1,1)}\right] = \frac{1}{2\pi i}\int_L \frac{\Gamma(\frac{n}{2}-\frac{s}{\alpha})\Gamma(1-\frac{s}{\alpha})}{\Gamma(1-s)}x^s ds \tag{38}$$

$$= \alpha\left\{\sum_{l=0}^{\infty}\frac{\Gamma(1-\frac{n}{2}-l)(-1)^l x^{\alpha(\frac{n}{2}+l)}}{\Gamma(1-\frac{an}{2}-\alpha l)(l)!} + \sum_{l=0}^{\infty}\frac{\Gamma(\frac{n}{2}-1-l)(-1)^l x^{\alpha(1+l)}}{\Gamma(1-\alpha-\alpha l)(l!)}\right\},$$

where n is not an even integer.

Thus for $n = 1$, we find that

$$N(x,t) = \frac{1}{2t^{\frac{\alpha}{2}}}\sum_{l=0}^{\infty}(-1)^l\frac{A^{\frac{l}{2}}}{\Gamma(1-\alpha(l+1)/2)(l!)}, \tag{39}$$

where $A = \frac{x^2}{t^\alpha}$ and the duplication formula for the gamma function is used.

For $n = 2$, the H-function of (37) is singular and in this case, the result is explicitly given by Barkai (2001) in the form

$$N(x,t) \sim \frac{1}{\pi\Gamma(1-\alpha)t^\alpha}ln[\frac{t^{\frac{\alpha}{2}}}{x}]. \tag{40}$$

For $n = 3$, the series expansion is given by

$$N(x,t) = \frac{1}{4\pi t^{3\alpha/2}A^{1/2}}\sum_{l=0}^{\infty}\frac{(-1^l)A^{l/2}}{\Gamma[1-\alpha(1+l/2)]}. \tag{41}$$

From above, it readily follows that for $n = 3$ and $\alpha \neq 1$

$$N(x,t) \sim \frac{1}{x}, \text{ as } x \to \infty. \tag{42}$$

8

It will not be out of place to mention that the one sided Lévy stable density can be obtained from Laplace inversion formula (31) by virtue of the identity

$$K_{\pm\frac{1}{2}}(x) = \left(\frac{\pi}{2x}\right)^{\frac{1}{2}} e^{-x}, \tag{43}$$

and can be conveniently expressed in terms of the Laplace transform

$$\int_0^\infty e^{-ut}\Phi_\rho(t)dt = e^{-u^\rho}, \ \ Re(u) > 0, \ Re(\rho) > 0. \tag{44}$$

The result is

$$\Phi_\rho(t) = \frac{1}{\rho}H_{1,1}^{1,0}\left[\frac{1}{t}\left|{(1,1) \atop (\frac{1}{\rho},\frac{1}{\rho})}\right.\right], \ (\rho > 0). \tag{45}$$

Note 2. This result is obtained earlier by Schneider and Wyss (1989) by following a different procedure. Asymptotic behavior of $\Phi_\alpha(t)$ is also given by Schneider (1986).

In conclusion, we mention that some of the results derived in this article may find some applications in problems associated with models of long-memory processes driven by Lévy noise and other related problems, see the article by Anh, Heyde, and Leonenko (2002).

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